Mathematical Formalism for Voting Process

Speaker: ***Alex Petrov, BitFury Group***

September 2015

Voting process

Voting is one of approaches to reach consensus in a decentralized environment (e.g., Bitcoin).

Problems of existing voting models (e.g., *x*-th percentile voting) :
- not all votes are taken into account
- some votes are given extraordinary weight, which theoretically allows *x*% attacks
- parameters of the voting process are not always grounded

Our proposal: rigorous mathematical model allowing to
- determine voting results algorithmically (with either an explicit expression or simple computational methods)
- take all votes into consideration
- maximize *all* voters' satisfaction with the voting result.

Additionally, our model is tunable and can be implemented for other voting processes in Bitcoin and beyond.

Function

Lets maximize all voters satisfaction with the voting result
via defining dissatisfaction function *F(s, v)*,
where

- *s* is the target parameter chosen in voting (e.g., block size limit)
- *v* is the vote

Suppose we have votes $v_1, v_2, \ldots, v_n$.

$$L(s) = \sum_{i=1}^{n} w_i F(s, v_i) \to \min_s$$

where w_i are (optional) vote weights.

Then, we can solve for target *s* by differentiating:

$$L'(s) = 0$$

Two types of dissatisfaction functions:

BitFury

$$F(s,v) = D(s - v)$$

A) absolute difference between a vote and the target

V \ S	1	2	4
1 mb	*D(0)*	*D(1)*	*D(3)*
2 mb	*D(-1)*	*D(0)*	*D(2)*
3 mb	*D(-2)*	*D(-1)*	*D(1)*
4 mb	*D(-3)*	*D(-2)*	*D(0)*
5 mb	*D(-4)*	*D(-3)*	*D(-1)*

$$L(s=1) = \sum_{i=1}^{5} D(1 - v_i)$$

$$F(s,v) = D(s/v - 1)$$

B) relative difference

V \ S	1	2	4
1 mb	*D(0%)*	*D(+100%)*	*D(+300%)*
2 mb	*D(-50%)*	*D(0%)*	*D(+100%)*
3 mb	*D(-67%)*	*D(-33%)*	*D(+33%)*
4 mb	*D(-75%)*	*D(-50%)*	*D(0%)*
5 mb	*D(-80%)*	*D(-60%)*	*D(-20%)*

$$L(s=2) = \sum_{i=1}^{5} D(\frac{2}{v_i} - 1)$$

- *V* miners vote
- *S* voting result

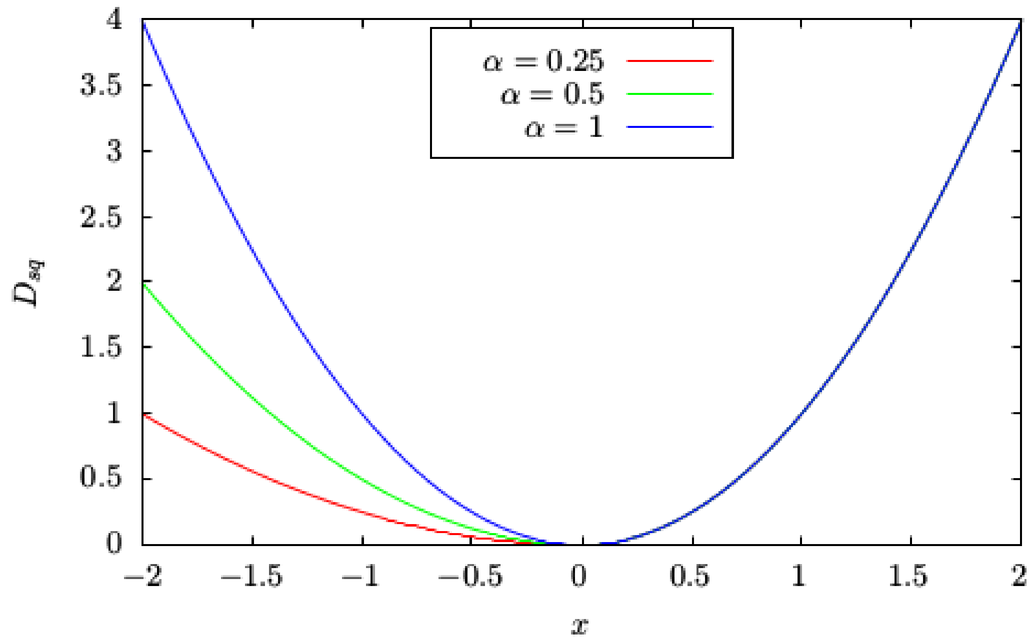
* *D* should be a non-negative, continuously differentiable, convex function with the only zero at 0.

Example

Example: quadratic dissatisfaction function

$$D_{sq}(x) = \begin{cases} x^2, & \text{if } x \geq 0, \\ \alpha x^2, & \text{if } x < 0. \end{cases}$$

$$F(s,v) = \{1, \alpha\}(s-v)^2$$



Example of votes distribution

Vote, Mb	1	2	4	8
Weight	21%	25%	25%	29%

Block-size result with α …

α	1	0.5	0.25	0.125
Vote result, Mb	4.03	3.25	2.59	2.13

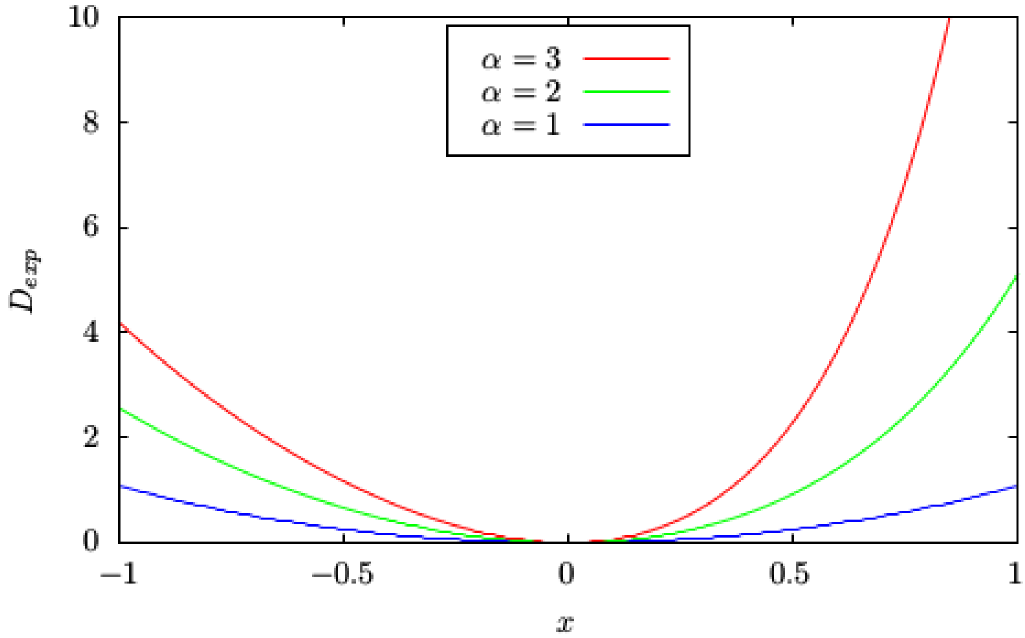
Parameter α allows to select how much the voting result will be skewed:

- α = 1 corresponds to a weighted average of the votes
- α < 1 skews the target to lower values
- α > 1 skews the target to higher values

Example

Example: exponential dissatisfaction function

$$D_{exp}(x) = e^{\alpha x} + \alpha e^{-x} - \alpha - 1$$



Example of votes distribution

Vote, Mb	1	2	4	8
Weight	21%	25%	25%	29%

Resulting block-size with α…

α	1	2	3	4	5
Vote result, Mb	4.47	3.40	2.82	2.46	2.22

Again, α determines skewness. Unlike the quadratic function, exponential function is infinitely continuously differentiable; however, optimization task requires more complex computations.

Thank You

BitFury

Calculating voters' dissatisfaction for the quadratic function with skew parameter α = 0.5. Total dissatisfaction is the weighted sum of voters' dissatisfactions.

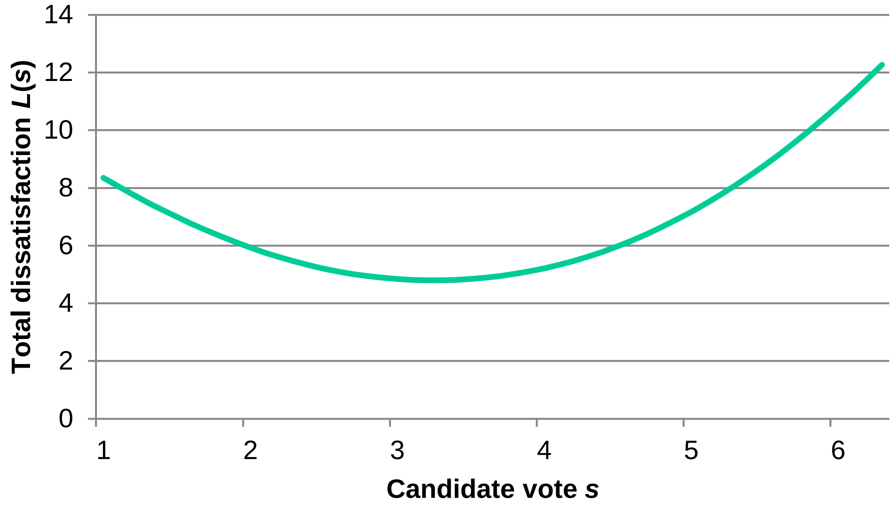
=(8 – 1)²

	weight w_i	vote v_i, MB	Candidate vote result, MB				Result
			1	2	4	8	**3.25**
Weighted voter's dissatisfaction	21%	1	0	1	9	49	**5.05**
	25%	2	0.5	0	4	36	**1.56**
	25%	4	4.5	2	0	16	**0.28**
	29%	8	24.5	18	8	0	**11.3**
Total dissatisfaction			8.36	5.93	5.21	23.29	**4.80**

See calculations on the next slide

= 0.5 (1 – 4)²

= 21% · 9 + 25% · 4 + 25% · 0 + 29% · 8



The optimal vote result 3.25 can be obtained by solving $L'(s) = 0$ on intervals (1, 2), (2, 4) and (4, 8).

On interval (1, 2):

$$L(s) = 0.21(s-1)^2 + 0.25 \cdot 0.5(s-2)^2 + 0.25 \cdot 0.5(s-4)^2 + 0.29 \cdot 0.5(s-8)^2 = 0.605s^2 - 4.24s + 11.99;$$
$$L'(s) = 1.21s - 4.24 = 0; \quad s = 4.24/1.21 \approx 3.504.$$

On interval (2, 4):

$$L(s) = 0.21(s-1)^2 + 0.25(s-2)^2 + 0.25 \cdot 0.5(s-4)^2 + 0.29 \cdot 0.5(s-8)^2 = 0.73s^2 - 4.74s + 12.49;$$
$$L'(s) = 1.46s - 4.74 = 0; \quad s = 4.74/1.46 \approx 3.247.$$

On interval (4, 8):

$$L(s) = 0.21(s-1)^2 + 0.25(s-2)^2 + 0.25(s-4)^2 + 0.29 \cdot 0.5(s-8)^2 = 0.855s^2 - 5.74s + 14.49;$$
$$L'(s) = 1.71s - 5.74 = 0; \quad s = 5.74/1.71 \approx 3.357.$$

Only the second value 3.247 is within the corresponding interval (2, 4), so we may skip checking other two values.